$\psi/(\nu_\infty ULc)^{1/2}$, and $\rho^* = \rho/\rho_\infty$ for which (8) and (9) become

$$u^* \frac{\partial u^*}{\partial x^*} = u^* \frac{\partial}{\partial \psi^*}\left(u^* \frac{\partial \mu^*}{\partial \psi^*}\right) + b\theta \tag{10}$$

$$\frac{\partial \theta}{\partial \psi_*} = \frac{1}{P}\frac{\partial}{\partial \psi^*}\left(u^* \frac{\partial \theta}{\partial \psi^*}\right) \tag{11}$$

when $b = aL/U^2$ (L being as yet arbitrary)

Let us assume a set of solutions of the form

$$u^* = Ax^{*p}f'(\eta) \qquad f(\eta) = \frac{\psi^*}{x^*q} \qquad \theta = Bx^{*\gamma}\xi(\eta)$$

Equations (10) and (11) become

$$A^2x^{*2p-1}[pf'^2 - qff''] = A^3x^{*3p-2q}f''' + Bbx^{*\gamma}\xi \tag{12}$$

$$Bx^{*\gamma-1}[\gamma f'\xi - q\xi'f] = \frac{AB}{P\gamma}x^{*p-2q+\gamma}\xi'' \tag{13}$$

If we now assume $p = \frac{1}{2}, \gamma = 0, q = \frac{3}{4}, A = \frac{1}{4}, B = 1, b = A^3$, giving $L = \frac{1}{64}(u^2/a)$, then Eqs (12) and (13) take the form

$$f''' + 3ff'' - 2f'^2 + \xi = 0 \tag{14}$$

$$\xi'' + 3P_rf\xi' = 0 \tag{15}$$

The boundary conditions now become, $f = f' = 0$, $\xi = 1$ for $\eta = 0$, and $f' = 0$, $\xi = 0$, $\eta = \infty$ Equations (14) and (15) are exactly identical with those deduced by Pohlhausen when he discussed the flow of a liquid along a hot vertical plate Taking buoyancy forces into consideration, the curves showing the solution already have been given by the forementioned author for different values of P Thus it is shown that the solution of the problem of compressible flow may be reduced to that for the problem of incompressible flow which now takes the form $n^* = \frac{1}{4}n^{*1/2}f'(\eta)$ $f(\eta) = \psi^*n^{*3/4}$, $\theta = \xi(\eta)$

Heat Transfer

The quantity of heat transfer per unit time and area is

$$q(x) = -k\left(\frac{\partial T}{\partial y}\right) = -\frac{k\rho u}{\rho_\infty}\left(\frac{\partial T}{\partial \psi}\right) = -\frac{k\rho u(T_\omega - T_\infty)}{\rho_\infty(\nu_\infty ULC)^{1/2}}\frac{\partial \theta}{\partial \psi^*}$$

$$= -\frac{k\rho u^* U(T_\omega - T_\infty)}{\rho_\infty(\nu_\infty ULC)^{1/2}}\frac{1}{f'(\eta)x^{3/4}}\left(\frac{\partial \theta}{\partial \eta}\right)$$

$$= -\frac{k\rho U(T_\omega - T_\infty)}{4\rho_\infty(\nu_\infty ULC)^{1/2}}\left(\frac{\partial \theta}{\partial \eta}\right)x^{*-1/4}$$

The quantity of heat transfer per unit time and area from the plate to the fluid at a distance x

$$= -\frac{k_0\rho_0(T_\omega - T_\infty)}{4\rho_\infty(\nu_\infty ULC)^{1/2}}UL^{1/4}\left(\frac{\partial \theta}{\partial \eta}\right)_0 x^{-1/4}$$

$$= -\frac{k_0}{4}\frac{(T_\omega - T_\infty)}{(\nu_0)^{1/2}}\frac{U^{1/2}}{L^{1/4}}\left(\frac{\partial \theta}{\partial \eta}\right)_0 x^{-1/4}$$

$$\left[\frac{\mu}{\mu_\infty} = C\frac{T}{T_\infty} = C\frac{\rho_\infty}{\rho}; \text{ therefore } \left(\frac{\nu}{\nu_\infty}\right)^{1/2} = c^{1/2}\frac{\rho_\infty}{\rho}\right]$$

Total heat transfer by a plate of length l and width b is

$$Q = b\int_0^L a(x)dx = -\frac{bk_0}{3}(T_\omega - T_\infty)\left(\frac{\partial \theta}{\partial \eta}\right)_0\left(\frac{UL}{\nu_0}\right)^{1/2}$$

$$= -\frac{bk_0}{3}(T_\omega - T_\infty)\left(\frac{\partial \theta}{\partial \eta}\right)_0 R_0^{1/2} \tag{16}$$

where $(\partial\theta/\partial\eta)_0$ depends upon Prandtl's number The mean Nusselt number is defined by

$$Q = bk_0(T_\omega - T_\infty)Nm$$

$$Nm = -\frac{1}{3}\left(\frac{\partial \theta}{\partial \eta}\right)_0 R_0^{1/2} \tag{17}$$

It should be noted that Q as well as the Nusselt number depends upon the Reynolds number and Prandtl number as usual

$$\tau(x) = \mu\left(\frac{\partial u}{\partial y}\right) = \frac{\mu\rho u}{\rho_\infty}\left(\frac{\partial u}{\partial \psi}\right) = \frac{\mu\rho u^* L^2}{\rho_\infty(\nu_\infty ULC)^{1/2}}\frac{1}{f'(\eta)x^{*3/4}}\left(\frac{\partial u^*}{\partial \eta}\right)$$

Local skin friction is $\tau_0(x) = [(\mu_0\rho_0U^3)^{1/2}/16L^{3/4}]f''(0)x^{1/4}$ The total value of skin friction over a portion of the plate of width b and length L is D_f:

$$D_f = b\int_0^L \tau_0(x)dx = \frac{b(\rho_0\mu_0U^3)^{1/2}}{20}L^{1/2}f''(0) \tag{18}$$

References

[1] Chapman, D R and Rubesin, M W, Temperature and velocity profiles in the compressible laminar boundary layer with arbitrary distribution of surface temperature," J Aerospace Sci 16, 547 (1949)

[2] Pohlhausen, E, 'Der Warmeaustausch Zwischen festen Korpern und Flussigkeiten mit Kleiner Reibung und Kleiner Warmeleitung," Z Angew Math Mech 1, 121 (1921)

[3] Von Mises, R, Bemenkung Zur Hydrodynamik," Z Angew Math Mech 7 (1927)

[4] Beckmann and Schmidt, 'Das temperature und Geschwindig Keibfeld von einer Warmealgeben den senKrechten Platte bei naturlicher Konvektion," Forsch Ingr Wiss 1391 (1940)

General Bibliography

Lees, L and Lin, C C, ' Investigation of the stability of the laminar boundary layer in a compressible fluid," NACA TN 1115 (1946)

Lees, L, ' The stability of the laminar boundary layer in a compressible fluid," NACA TN 1360 (1947)

Ostrach, S, An analysis of laminar free convection flow and heat transfer about a flat plate parallel to the direction of the generating body force," NACA Rept 1111 (1953)

Schlighting, H, *Boundary Layer Theory* (McGraw Hill Book Co Inc, New York, 1960) 4th ed p 332

Exact First-Order Navigation-Guidance Mechanization and Error Propagation Equations for Two-Body Reference Orbits

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Nomenclature

r	=	radial distance to dynamical center
p	=	semilatus rectum
n^*	=	$(\mu/p^3)^{1/2}$ = modified mean motion parameter
v	=	true anomaly
e	=	eccentricity
τ	=	$t - t_0$ = time since initial epoch
Δx	=	inertial horizontal in-plane position error
Δy	=	inertial out-of-plane position error

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Table 1 Summary of six-parameter sensitivity coefficients for two-body reference orbits (exact first order, inverse)

Δ	[illegible]		[illegible]	[illegible]		[illegible]	
		[illegible]			[illegible]		[illegible]
	( )		[illegible]	[illegible]		[illegible]	
ẋ	[illegible]		[illegible]	[illegible]		( )	Δẋ
Δẏ*		[illegible]			[illegible]		ẏ
Δż	[illegible]		[illegible]	[illegible]		[illegible]	ż
Δ	[illegible]		[illegible]	[illegible]		[illegible]	
		[illegible]			[illegible]		
	( )		[illegible]	[illegible]		[illegible]	
ẋ	[illegible]		[illegible]	[illegible]		( )	ẋ
ẏ		[illegible]			[illegible]		ẏ
ż	[illegible]		[illegible]	[illegible]		[illegible]	ż

Δz = inertial radial position error
$\Delta \dot{x}^*$ = inertial horizontal in-plane (circumferential) velocity error
$\Delta \dot{y}^*$ = inertial out-of-plane velocity error
$\Delta \dot{z}^*$ = inertial radial velocity error
μ = gravitational constant of the primary body
ρ = p/r = nondimensional = $1 + e \cos v$
cv, sv = $\cos v$, $\sin v$

Introduction

THIS note presents an exact first-order perturbation solution and an associated linear guidance mechanization for any conical reference orbit A number of papers (Refs 1–3 and 5) have appeared in the literature dealing with this particular problem for circular and nearly circular orbits It is believed, however, that an exact explicit representation for all two-body orbits has not been completely treated in the current literature

Results

Table 1 presents a complete exact first-order solution to the two body relative motion (linear perturbation) equations of motion This solution which is called the transition matrix, constitutes the complete six parameter solution and is valid for all two-body terminals Table 2 is an exact explicit first-order guidance law mechanization for impulsive controls which is valid for all conical orbits The results in Table 1, in particular, have been checked by two essentially different derivations One derivation involved perturbing the integrals of the two-body solution This derivation is given in Ref 5 The other involved integrating the two-body perturbation equations of motion An outline of the latter, which is a summary of Ref 6, is discussed herein

The form of the solution as given in Table 1 applies directly to circular, elliptical, and hyperbolic orbits without modification For the parabolic and nearly parabolic orbits, however, it is necessary to expand this solution about unity with respect to the eccentricity parameter e, using a modified form of Kepler's equation This expansion is required to minimize the effect of roundoff errors if machine computation is contemplated The velocity-to-be-gained sensitivity matrices (Table 2), on the other hand, apply to all conics

Properties of Solution

The inverse of the matrix (lower matrix of Table 1) is simply obtained by reversing time That is, to generate the inverse, the subscripts between the two terminals are interchanged This property follows from physical reasoning as well as from considering the integration of the adjoint perturbation equations backward in time [5] The property of obtaining the inverse by transposition of the submatrices, as given in Ref 8, is also seen to hold for local level coordinates

The terminal rendezvous or the relative motion equations can be obtained from Table 1 upon replacing the inertial velocity deviations (designated by asterisks) by time differentials of the corresponding position deviations This will yield the "Clohessy-Wiltshire" equations for general conical orbits

Outline of Derivation

Details of the complete derivation will be given in a forthcoming paper A brief outline is given here, however, to show the specific relationship of the solution to the perturbation equations In a central inverse square field, the equations of motion of a satellitic point in an inertial coordinate system can be expressed as

$$\ddot{\mathbf{r}} = -\mu \mathbf{r}/r^3 \quad (1)$$

A first-order perturbation of Eq (1) referred to a rotating coordinate system yields the relative or the linear perturbation equation

$$\Delta\ddot{\mathbf{r}} - 2\omega\Delta\dot{\mathbf{r}} + (-\omega + \omega^2)\Delta\mathbf{r} = -\frac{\mu}{r^3}\left[I - \frac{3\mathbf{r}\mathbf{r}^T}{r^2}\right]\Delta\mathbf{r} \quad (2)$$

where $(\mathbf{r}\mathbf{r}^T/r^2)$ is a dyad or a tensor of rank 2, and ω is the angular velocity matrix of the rotating coordinate system Let the inertial perturbation of $\mathbf{r}$ be selected in the rotating frame as

Table 2 Explicit guidance law mechanization for two-body reference orbits (exact first order)

Velocity-To-Be-Gained Sensitivity Matrix of the First Kind $(-\beta^{-1}\alpha)$

$\frac{-n^*}{\nabla(\beta)}\left[\frac{3e\,sv_o}{\rho}n^*\tau\left[s(v-v_o)+e\,sv-e\,sv_o(1+\rho)\right]+\frac{1}{\rho^2\rho_o}\left[(2\rho^3-\rho\rho_o^2)s(v-v_o)+2e(\rho_o sv_o-\rho sv)(-1+(1+\rho)c(v-v_o))\right]\right]$		$\frac{-n^*}{\nabla(\beta)}\left[3n^*\tau\frac{\rho_o}{\rho}\left[-s(v-v_o)-e\,sv+e\,sv_o(1+\rho)\right]+\frac{1}{\rho^2}\left[3(\rho-\rho_o)\rho+(\rho(1+\rho_o)+2\rho_o(1+\rho))(1-c(v-v_o))\right]\right]$
	$-n^*\rho_o\left[ctn(v-v_o)+\frac{e\,cv}{s(v-v_o)}\right]$	
$\frac{-n^*}{\nabla(\beta)}\left[3n^*\tau\frac{\rho_o}{\rho}\left[-s(v-v_o)-e\,sv+e\,sv_o(1+\rho)\right]+\frac{1}{\rho^2}\left[3(\rho-\rho_o)\rho+(\rho(1+\rho_o)+2\rho_o(1+\rho))(1-c(v-v_o))\right]\right]$		$\frac{-n^*}{\nabla(\beta)}\left[-3n^*\tau\frac{\rho_o}{\rho}\left[c(v-v_o)+e\,cv_o(1+\rho)\right]+\frac{1}{\rho^2\rho_o}\left[(2\rho_o^2+3\rho\rho_o^2-\rho(1+e^2))s(v-v_o)-2e\rho(sv-sv_o)\right]\right]$

Velocity To-Be-Gained Sensitivity Matrix of the Second Kind $(\beta^{-1})^T$

$\frac{n^*}{\nabla(\beta)}\left[-3e^2sv\,sv_o\,n^*\tau+\frac{1}{\rho\rho_o}\left[\rho\rho_o s(v-v_o)+2e(\rho_o sv_o-\rho sv)\right]\right]$		$\frac{n^*}{\nabla(\beta)}\left[3e\,sv\rho_o n^*\tau-\frac{1}{\rho\rho_o}\left[-\rho(1+\rho_o)c(v-v_o)+2\rho_o^2-\rho(\rho-1)\right]\right]$
	$n^*\rho\rho_o\frac{1}{s(v\ v_o)}$	
$\frac{n^*}{\nabla(\beta)}\left[3e\,sv_o\rho n^*\tau-\frac{1}{\rho\rho_o}\left[\rho_o(1+\rho)c(v-v_o)\ 2\rho^2+\rho_o(\rho_o-1)\right]\right]$		$\frac{n^*}{\nabla(\beta)}\left[-3\rho\rho_o n^*\tau+\frac{1}{\rho\rho_o}\left[(1+\rho)(1+\rho_o)s(v-v_o)+e(1+\rho)sv-e(1+\rho_o)sv_o\right]\right]$

$$\mathrm{Det}\,\beta = \nabla(\beta) = \frac{-3n^*\tau}{\rho\rho_o}\left[(1+e^2)s(v-v_o)+2e(sv-sv_o)\right]+\frac{2}{\rho^2\rho_o^2}\left[(\rho(1+\rho_o)+\rho_o(1+\rho))(1-c(v-v_o))+(\rho\ \rho_o)^2\right]$$

$$\Delta r = \begin{bmatrix}\Delta x\\ \Delta y\\ \Delta z\end{bmatrix} = \begin{bmatrix}\text{local inertial horizontal in-plane component of error}\\ \text{local inertial out-of-plane component of error}\\ \text{local inertial vertical component of error}\end{bmatrix}$$

Assuming the reference trajectory is Keplerian, Eq (2) can be reduced in terms of the in-plane, out-of-plane components, to yield the following equations:

$$\Delta\ddot{x} - n^{*2}\frac{p^3}{r^3}\left(\frac{p}{r}-1\right)\Delta x + 2n^*\frac{p^2}{r^2}\Delta\dot{z} - 2n^{*2}\frac{p^3}{r^3}e\,sv\,\Delta z = 0 \text{ (normal in-plane)}$$

$$\Delta\ddot{y} + n^{*2}\frac{p^3}{r^3}\Delta y = 0 \qquad \text{(out-of-plane)}$$

$$\Delta\ddot{z} - n^{*2}\frac{p^3}{r^3}\left(\frac{p}{r}+2\right)\Delta z - 2n^*\frac{p^2}{r^2}\Delta\dot{x} + 2n^{*2}\frac{p^3}{r^3}e\,sv\,\Delta x = 0 \text{ (radial)} \quad (3)$$

The corresponding inertial velocity error (or deviation) referred to the rotating coordinate system is given by

$$\Delta\dot{\mathbf{r}}^* \equiv \begin{bmatrix} \Delta\dot{x}^* \\ \Delta y^* \\ \Delta\dot{z}^* \end{bmatrix} = \begin{bmatrix} \Delta\dot{x} + n^*(p^2/r^2)\Delta z \\ \Delta\dot{y} \\ \Delta\dot{z} - n^*(p^2/r^2)\Delta x \end{bmatrix} \quad (4)$$

Solving Eqs (3) and (4) as an initial value problem will yield the "transition" matrix, which is the desired solution as given in Table 1 It is needless to state here that the derivation for this solution is complex inasmuch as the differential equations involve time-varying coefficients An exact integral to Eq (3) has been obtained but will not be given here for lack of space

For circular orbits $p/r = 1$, and Eqs (3) and (4) become constant coefficient equations which can be directly solved as an initial value problem using Laplace Transform techniques This solution is given in Refs 1–5

Let the matrix solution of Table 1 be denoted by ϕ and assume that it is partitioned as a set of four 3×3 submatrices Then the solution to Eq (3) can be represented as

$$\begin{bmatrix} \Delta\mathbf{r}_{n+1} \\ \Delta\dot{\mathbf{r}}_{n+1}^* \end{bmatrix} = \begin{bmatrix} \alpha_{n+1} & \beta_{n+1\ n} \\ \dot{\alpha}_{n+1\ n} & \dot{\beta}_{n+1\ n} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{r}_n \\ \Delta\dot{\mathbf{r}}_n^* \end{bmatrix} \quad (5)$$

in which this matrix, defined as $\phi_{n+1\ n}$, is given by Table 1, and where, typically, $\alpha_{n+1\ n} = \alpha(t_{n+1}, t_n)$ is a 3×3 submatrix evaluated between the two terminals $n + 1$ and n The inertial position and velocity deviation propagation and error equations are then given by Eq (5) To obtain the inverse (lower matrix of Table 1), use can be made of the relationship

$$\phi^{-1} = \begin{bmatrix} \beta^T & -\dot{\beta}^T \\ -\dot{\alpha}^T & \alpha^T \end{bmatrix} \quad (6)$$

which is proved in Ref 8 for inertial coordinates systems and in Refs 6 and 7 for rotating coordinate systems

An Explicit Linear Guidance Law

Using Eq (5), a linear deterministic solution to the two-point boundary problem that involves impulsive controls can be developed In most linear guidance schemes in the free-fall regime, it is desired to secure a position and/or velocity match at a future epoch This implies one velocity correction to null the predicted position deviation and a second correction to null the velocity deviation The velocity-to-be-gained to null the predicted position deviation at epoch $n + 1$ is given as

$$\Delta\mathbf{v}_n{}^1 = -[\beta_{n+1\ n}{}^{-1}\ \alpha_{n+1\ n} \vdots \mathrm{I}] \begin{bmatrix} \Delta\mathbf{r}_n \\ \Delta\dot{\mathbf{r}}_n^* \end{bmatrix} \quad (7)$$

The velocity-to-be-gained to be applied at $n + 1$ to null the velocity deviation at $n + 1$ is given by

$$\Delta_{n+1}\mathbf{v}^2 = (\beta_{n+1\ n}{}^{-1})^T \Delta\mathbf{r}_n \quad (8)$$

Further details on linear impulsive guidance mechanization techniques are given in Ref 7 The previously cited Table 2 yields the explicit representation of the velocity-to-be-gained sensitivity matrixes, $-\beta^{-1}\alpha$ and $[\beta^{-1}]^T$, for all two-body conical orbits

References

[1] Clohessy, W H and Wiltshire, R S, "Terminal guidance system for satellite rendezvous," J Aerospace Sci **27**, 653–658 674 (1960)

[2] Eggleston, J M and Dunning, R S, "Analytical evaluation of a method of midcourse guidance for rendezvous with earth satellites" NASA TN D-883 (October 1962)

[3] Hord, R A, "Relative motion in the terminal phase of interception of a satellite or ballistic missile" NASA TN 4399 (1958)

[4] Wheelon, A D, "Midcourse and terminal guidance," *Space Technology* (John Wiley and Sons, Inc, New York, 1959), Chap 26

[5] Kochi, K C, "An introduction to midcourse navigation-guidance," Autonetics Rept EM 262-265 (October 1962)

[6] Kochi, K C, "Navigation-guidance mechanization equations for two-body orbits," Autonetics Tech Memo 243-2-150 (May 1963)

[7] Kochi, K C, "A 'universal' linear guidance mechanization for space systems," Autonetics Tech Memo 243-2-153 (June 1963)

[8] McLean, J D, Schmidt, S F, and McGee, L A, "Optimal filtering and linear prediction applied to a midcourse navigation system for circumlunar missions," NASA TN D-1208 (1962)

Mars Nonstop Round-Trip Trajectories

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Introduction

MANNED and unmanned nonstop round trips may be precursors of the first manned landing on Mars The vehicle for such a trip will be launched from Earth and, without stopping, fly by Mars and return to Earth Except in special cases, a vehicle placed on a trajectory to Mars will not return to Earth In many cases, however, an Earth return can be achieved if the trajectory is modified by one of the following means: 1) by the gravity of Mars, 2) by gravity supplemented by propulsion, or 3) by gravity supplemented by aerodynamic forces The analysis of the trajectories resulting from the latter two trajectory modifications is the contribution made herein

In this note, the preceding three kinds of nonstop round trip trajectories are compared in the years 1971 and 1980 on the basis of mission time and the required propulsive velocity increment A short mission is desirable for reducing life support system weight and for psychological reasons Since the initial system gross weight is exponentially related to the propulsive velocity increment, a low value of this parameter also is desired

Types of Maneuvers

The trajectory of a typical nonstop round-trip mission in 1971 is superimposed on orbits of Mars and Earth in Fig 1 Note that the Mars orbit is quite eccentric The vehicle leaves Earth at point 1, passes close to Mars at point 2, and returns to Earth at point 3 In general, outbound and in bound legs of the trajectory are of unequal length

When the vehicle passes Mars, its trajectory can be changed in three ways, which characterize the three trajectories analyzed (Fig 2) For a gravity turn, only the gravitational field of Mars deflects the vehicle (Fig 2a) The arrival and the departure velocities V_A and V_D are equal in magnitude, and the turning is equally distributed between the arrival and the departure phases of the maneuver, that is, $\Phi_{GA} = \Phi_{GD}$

If gravity alone cannot produce an Earth return trajectory, the trajectory can be further changed by thrusting The thrust is generally best applied at the sphere of influence when the vehicle departs from Mars Figure 2a illustrates a velocity increment $\Delta V_{♂}$ imparted to change the departure velocity vector from V_D to V_D A trajectory using this maneuver is called a propulsive-gravity turn

An aerodynamic-turn trajectory (Fig 2b) may be used to advantage when a low departure velocity and a high turning angle are required The total turning can be broken into three stages First, the arrival velocity vector is turned through an angle Φ_{GA} by the planetary gravitation field Second, after entering the Mars atmosphere, the vehicle

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